Torus

Eric W. Weisstein*

An (ordinary) torus is a surface having genus one, and therefore possessing a single "hole" (left figure 1). The single-holed "ring" torus is known in older literature as an "anchor ring." It can be constructed from a rectangle by gluing both pairs of opposite edges together with no twists (right figure 1; for details see [3] and [5]). The usual torus embedded in three-dimensional space is shaped like a donut, but the concept of the torus is extremely useful in higher dimensional space as well.

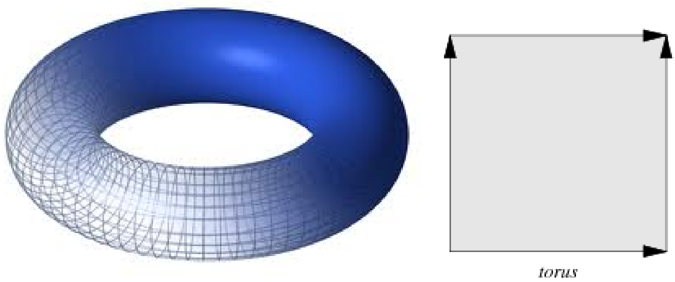


Figure 1: Torus

In general, tori can also have multiple holes, with the term n-torus used for a torus with n holes. The special case of a 2-torus is sometimes called the double torus, the 3-torus is called the triple torus, and the usual single-holed torus is then simple called "the" or "a" torus.

A second definition for n-tori relates to dimensionality. In one dimension, a line bends into circle, giving the 1-torus. In two dimensions, a rectangle wraps to a usual torus, also called the 2-torus. In three dimensions, the cube wraps to form a 3-manifold, or 3-torus. In each case, the n-torus is an object that exists in dimension $n + 1$. One of the more common uses of n-dimensional tori is in dynamical systems. A fundamental result states that the phase space trajectories of a Hamiltonian system with n degrees of freedom and possessing n integrals of motion lie on an n-dimensional manifold which is topologically equivalent to an n-torus (see [10]).

Torus coloring of an ordinary (one-holed) torus requires 7 colors, consistent with the Heawood conjecture.

Let the radius from the center of the hole to the center of the torus tube be c, and the radius of the tube be a. Then the equation in Cartesian coordinates for a torus azimuthally symmetric about the z-axis is

*From MathWorld–A Wolfram Web Resource http://mathworld.wolfram.com/Torus.html

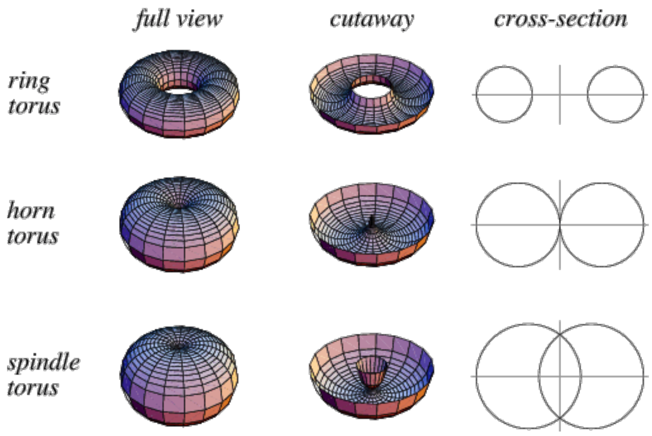
$$(c - \sqrt{x^2 + y^2})^2 + z^2 = a^2 \tag{1}$$

and the parametric equations are

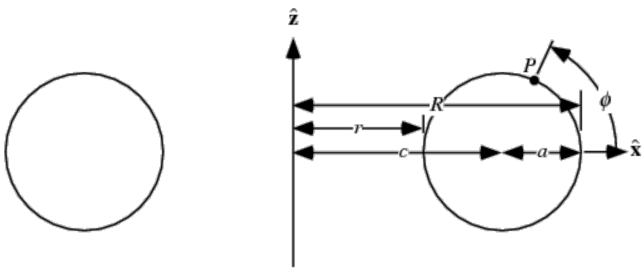
$$\begin{aligned} x &= (c + a\cos v)\cos u \\ y &= (c + a\cos v)\sin u \\ z &= a\sin v \end{aligned} \tag{2}$$

for $u, v \in [0, 2\pi)$. Three types of torus, known as the standard tori, are possible, depending on the relative sizes of a and c: $c > a$ corresponds to the ring torus (shown above), $c = a$ corresponds to a horn torus which is tangent to itself at the point $(0, 0, 0)$, and $c < a$ corresponds to a self-intersecting spindle torus (see [8]).

If no specification is made, "torus" is taken to mean ring torus. The three standard tori are illustrated below, where the first image shows the full torus, the second a cut-away of the bottom half, and the third a cross section of a plane passing through the z-axis.



The standard tori and their inversions are cyclides. If the coefficient of $\sin v$ in the formula for z is changed to $b \neq a$, an elliptic torus results.



To compute the metric properties of the ring torus, define the inner and outer radii by

$$r \equiv c - a, \qquad R \equiv c + a \tag{3}$$

Solving (3) for a and c gives

$$a = \frac{1}{2}(R - r), \qquad c = \frac{1}{2}(R + r).$$

Then the surface area of this torus is

$$S = (2\pi a)(2\pi c) = 4\pi^2 ac = \pi^2(R^2 - r^2)$$

and the volume can be computed from Pappus's centroid theorem

$$V = (\pi a^2)(2\pi c) = 2\pi^2 a^2 c = \frac{1}{4}\pi^2(R - r)^2(R + r).$$

The volume can also be found by integrating the Jacobian computed from the parametric equations (2) of the solid,

$$\begin{aligned} x &= (c + r' \cos v)\cos u \\ y &= (c + r' \cos v)\sin u \\ z &= r' \sin v, \end{aligned}$$

which simplifies to

$$J = \left| \frac{\partial(x, y, z)}{\partial(u, v, r')} \right| = r'(c + r' \cos v),$$

giving

$$V = \int_0^{2\pi} \int_0^{2\pi} \int_0^{a} r'(c + r' \cos v) dr' du dv = 2\pi^2 a^2 c$$

as before.

The moment of inertia tensor of a solid torus with mass M is given by

$$I = \begin{pmatrix} (\frac{5}{8}a^2 + \frac{1}{2}c^2)M & 0 & 0 \\ 0 & (\frac{5}{8}a^2 + \frac{1}{2}c^2)M & 0 \\ 0 & 0 & (\frac{3}{4}a^2 + c^2)M \end{pmatrix}$$

The coefficients of the first fundamental form are

$$E = (c + a\cos v)^2, \; F = 0, \; G = a^2$$

and the coefficients of the second fundamental form are

$$e = -(c + a\cos v)\cos v, \; f = 0, \; g = -a,$$

giving:

- Riemannian metric

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 dv^2$$

- area element

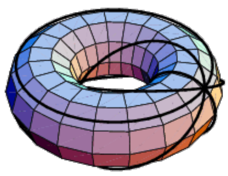
$$dA = a(c + a\cos v) du \wedge dv$$

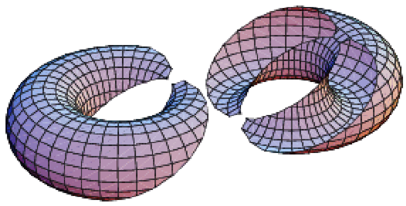
(where $du \wedge dv$ is a wedge product)

- Gaussian and mean curvatures

$$K = \frac{\cos v}{a(c + a\cos v)}, \quad H = -\frac{c + 2a\cos v}{2a(c + a\cos v)}.$$

A torus with a hole in its surface can be turned inside out to yield an identical torus. A torus can be knotted externally or internally, but not both. These two cases are ambient isotopies, but not regular isotopies. There are therefore three possible ways of embedding a torus with zero or one knot.





An arbitrary point P on a torus (not lying in the xy-plane) can have four circles drawn through it. The first circle is in the plane of the torus and the second is perpendicular to it. The third and fourth circles are called Villarceau circles (see [11, 9, 2, 7]).

References

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